Combinatorial interpretation of Haldane-Wu fractional exclusion statistics

A.K. Aringazin and M.I. Mazhitov
Department of Theoretical Physics, Institute for Basic Research,
Eurasian National University, Astana 473021 Kazakstan

15 February 2002

Abstract

Assuming that the maximal allowed number of identical particles in state is an integer parameter, q, we derive the statistical weight and analyze the associated equation which defines the statistical distribution. The derived distribution covers Fermi-Dirac and Bose-Einstein ones in the particular cases $q = 1$ and $q \to \infty$ ($n_i/q \to 1$), respectively. We show that the derived statistical weight provides a natural combinatorial interpretation of Haldane-Wu fractional exclusion statistics, and present exact solutions of the distribution equation.

PACS 05.30Pr

1 Introduction

Statistics which are different from Fermi-Dirac and Bose-Einstein ones become of much interest in various aspects. A recent example is given by Haldane-Wu fractional exclusion statistics (FES) [1, 2] which is used to describe elementary excitations of a number of exactly solvable one-dimensional models of strongly correlated systems, and other models [2, 3]. This statistics is based on the statistical weight, which is a generalization of Yang-Yang [4] state counting as mentioned by Wu,

$$W_i = \frac{(z_i + (n_i - 1)(1 - \lambda))!}{n_i!(z_i - \lambda n_i - (1 - \lambda))!}, \tag{1}$$

where the parameter λ varies from $\lambda = 0$ (Bose-Einstein) to $\lambda = 1$ (Fermi-Dirac). This formula is a simple generalization and interpolation of Fermi and Bose statistical weights. While there is no physical meaning ascribed to λ here, the physical interpretation of Eq. (1) is that the *effective* number of available single-particle states *linearly* depends on the number of particles,

$$z_i^f = z_i - (1 - \lambda)(n_i - 1), \quad z_i^b = z_i - \lambda(n_i - 1), \tag{2}$$

for fermions and bosons, respectively. This is viewed as a defining feature of the fractional exclusion statistics.

In the present paper, we show that the equation which defines Haldane-Wu statistical distribution can be derived from a different statistical weight, which has a clear combinatorial and physical treatment. Also, we present exact solutions of this equation.

2 The combinatorics

A number of quantum states of n_i identical particles occupying z_i states, with up to q particles in state, $1 \leq q \leq n_i$, can be counted as follows.

We consider a configuration defined as that it has a maximal possible number of totally occupied states (exactly q particles in state). A number of such totally occupied states is an integer part of n_i/q which we denote by $\left[\frac{n_i}{q}\right]$. If q is a divisor of n_i we have identically $\left[\frac{n_i}{q}\right] = n_i/q$, so that the number of unoccupied states is $z_i - \frac{n_i}{q}$. If q is not a divisor of n_i we have one partially

occupied state, so that the number of unoccupied states is $z_i - \frac{n_i}{q} - 1$. We write a combined formula of the statistical weight for both the cases as

$$W_i = \frac{\left(z_i + n_i - \left[\frac{n_i}{q}\right]\right)!}{n_i!\left(z_i - \left[\frac{n_i}{q}\right] - l\right)!}, \tag{3}$$

where $l = 0$ or 1 if n_i/q is integer or noninteger, respectively; $i = 1, 2, \ldots, m$.

In the particular cases, $q = 1$ and $q = n_i$, we have $\left[\frac{n_i}{q}\right] = n_i/q$ and $l = 0$ so that Eq. (3) reduces to Fermi-Dirac and Bose-Einstein statistical weights, respectively,

$$W_i = \frac{z_i!}{n_i!(z_i - n_i)!}, \quad W_i = \frac{(z_i + n_i - 1)!}{n!(z_i - 1)!}. \tag{4}$$

As one can see, the effective number of available single-particle states derived from Eq. (3),

$$z_i^f = z_i - \left(1 - \frac{1}{q}\right) n_i, \quad z_i^b = z_i - \frac{1}{q} n_i + 1, \tag{5}$$

for fermions and bosons, respectively, is linear in n_i. With the identification of the parameters, $1/q = \lambda$, and the redefinition, $z_i \to z_i - (1 - \lambda)$, the statistical weight (3) coincides with Haldane-Wu statistical weight (1), for the case of integer n_i/q. Consequently, the obtained statistical weight (3) corresponds to a kind of fractional exclusion statistics. To verify whether (3) leads to Haldane-Wu distribution we obtain below the equation which governs statistical distribution.

3 The distribution function

Starting with Eq. (3), we follow usual technique of statistical mechanics to derive the associated most-probable distribution of n_i.

The thermodynamical probability is $W = \prod W_i$, and the entropy, $S = k \ln W$, can be calculated by using the approximation of big number of particles, $n! \simeq n^n e^{-n}$ for big n. Assuming conservation of the total number of particles, $N = \sum n_i$ and the total energy, $E = \sum n_i \varepsilon_i$, variational study of S corresponding to an equilibrium state gives us

$$\delta S = k \sum_i \Bigg[\left(1 - \frac{1}{q}\right) \ln \left(n_i + z_i - \frac{n_i}{q}\right) - \ln n_i$$

$$+\frac{1}{q}\ln\left(z_i-\frac{n_i}{q}\right)-\alpha-\beta\varepsilon_i\Big]\delta n_i=0, \tag{6}$$

where α and β are Lagrange multipliers, and we have used $\left[\frac{n_i}{q}\right]\simeq\frac{n_i}{q}$ and $l=0$ for big n_i. Using the notation $\kappa=1/q$ and inserting $\alpha=-\mu/kT$ and $\beta=1/kT$ (obtained via an identification of S, at $q=1$, with the thermodynamical expression), we rewrite Eq. (6) as

$$\frac{(z_i+(1-\kappa)n_i)^{1-\kappa}(z_i-\kappa n_i)^{\kappa}}{n_i}=\exp\frac{\varepsilon_i-\mu}{kT},\quad \kappa=1,\frac{1}{2},\frac{1}{3},\ldots \tag{7}$$

To draw parallels with Haldane-Wu statistics below we make analytic continuation of the discrete parameter κ assuming $\kappa\in[0,1]$. Under this condition, the derived distribution equation (7) *does reproduce* that of Haldane-Wu fractional exclusion statistics (Eq. (14) of ref. [2]), with $\kappa=\lambda$.

Below, we turn to consideration of properties and exact solutions of Eq. (7).

In general, Eq. (7) can not be solved exactly with respect to n_i. However, for $\kappa=1$ and $\kappa\to 0$ ($\kappa n_i\to 1$), it becomes linear in n_i and gives Fermi and Bose distributions, respectively. Also, we note that for $\kappa=1/2$, $1/3$, and $1/4$ the equation contains a polynomial of degree up to 4 so that it can be solved exactly for all these cases.

A convenient expression for n_i obeying Eq. (7) is given by [2]

$$n_i=\frac{1}{w(x)+\kappa}, \tag{8}$$

where we have redefined, $n_i/z_i\to n_i$, $x\equiv\exp[(\varepsilon_i-\mu)/kT]$, and the function $w(x)$ satisfies

$$(1+w)^{(1-\kappa)}w^{\kappa}=x. \tag{9}$$

Remarkably, exclusons which are "close" to fermions can be described in terms of exclusons which are "close" to bosons. In fact, we note that Eq. (7) is invariant under a set of transformations,

$$\kappa\to 1-\kappa,\quad n_i\to -n_i,\quad x\to -x, \tag{10}$$

for $\kappa\neq 0,1$. Therefore, if $n_i(x,\kappa)$ satisfies Eq. (7) then the function $m_i=-n_i(-x,1-\kappa)$ satisfies the same equation. Thus, we obtain the following general relation

$$n_i(-x,1-\kappa)=-n_i(x,\kappa),\quad \kappa\neq 0,1. \tag{11}$$

We see that the distribution n_i of exclusons for, e.g., $\kappa = 1/200 \simeq 0$ can be obtained from that of "dual" exclusons, with $\kappa = 1 - 1/200 = 199/200 \simeq 1$.

The values $\kappa = 1$ and $\kappa \to 0$ ($\kappa n_i \to 1$) are the only two points of *degeneration* of Eq. (7). Hence, any "deviation" from Fermi or Bose statistics is characterized by a sharp change of statistical properties, sending us to consideration of exclusons. Consequently, we can divide particles into three main types, genuine fermions, genuine bosons, and exclusons ($\kappa \in]0,1[$), since their statistical distributions obey *different non-degenerate* equations.

A fixed point of the map $\kappa \to 1-\kappa$ is $\kappa = 1/2$. Hence it represents a special case worth to be considered separately. In this case, Eq. (7) allows an exact solution and the result is (positive root) [2]

$$n_i = \frac{2}{\sqrt{1+4x^2}} = \frac{2}{\left(1 + 4\exp\frac{2(\varepsilon_i - \mu)}{kT}\right)^{1/2}}. \tag{12}$$

This distribution represents statistics with up to two particles in state, $q = 2$ (semions).

We have obtained exact solutions (real roots) of Eq. (7) for $\kappa = 1/3$ and $2/3$ which we write as

$$n_i = \frac{3}{f + f^{-1} \mp 1}, \tag{13}$$

where

$$f = \left[2\sqrt{y(y \mp 1)} + 2y \mp 1\right]^{1/3}, \quad y = 2\left(\frac{3x}{2}\right)^3 \pm 1. \tag{14}$$

From Eqs. (13) and (14) one can see how exclusons with $\kappa = 1/3$ (upper sign) are related to exclusons with $\kappa = 2/3$ (lower sign) that agrees with Eq. (11). Also, for $\kappa = 1/4$ and $3/4$ we have obtained the following exact solutions (positive real roots):

$$n_i = \frac{4}{\sqrt{2g^{-1/2} - g + 3} \pm g^{1/2} \mp 2}, \tag{15}$$

where

$$g = \frac{3}{2}\left([z^2(z+2)]^{1/3} + [z(z+2)^2]^{1/3}\right) + 1, \quad z = \sqrt{3\left(\frac{4x}{3}\right)^4 + 1} - 1. \tag{16}$$

Plots of $n_i(x)$ for various κ are presented in Fig. 1, from which one can see that these exclusons behave similar to fermions.

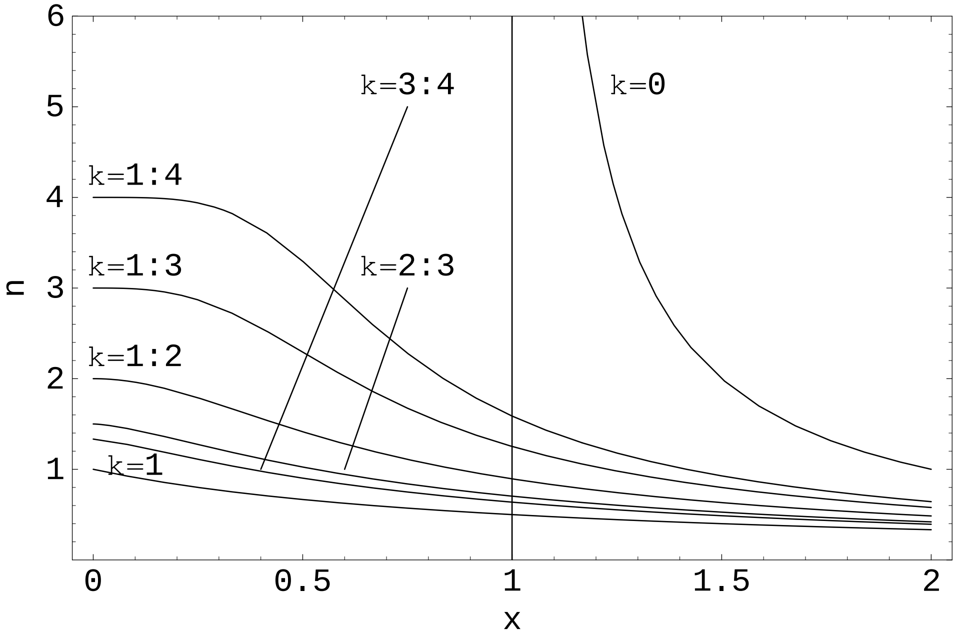

Figure 1: Statistical distribution n_i as a function of $x = \exp[(\varepsilon_i - \mu)/kT]$, for $\kappa = 1$ (fermions), $\kappa = 0$ (bosons), $\kappa = 1/2$ (semions, Eq. (12)), $\kappa = 1/3$ (Eq. (13), upper sign), $\kappa = 1/4$ (Eq. (15), upper sign), $\kappa = 2/3$ (Eq. (13), lower sign), and $\kappa = 3/4$ (Eq. (15), lower sign).

Distributions of exclusons can be obtained from a different approach, based on the canonical statistical sum which implies the mean number of particles,

$$n = \frac{\sum\limits_{N=0}^{q} N x^{-N}}{\sum\limits_{N=0}^{q} x^{-N}}. \tag{17}$$

This formula gives (exact) Fermi and Bose distributions for $q = 1$ and $q \to \infty$, respectively, while for arbitrary $q \geq 1$ the sum is

$$n = -\frac{x^{1+q} - (1+q)x + q}{(x^{1+q} - 1)(x - 1)}, \quad q = 1/\kappa. \tag{18}$$

In Fig. 2, we compare distributions (18) with exact solutions shown in Fig. 1. One can see that deviations become considerable as κ goes to smaller values.

However, we expect that near $\kappa = 0$ there should be a better correspondence since one approaches the other interpolation endpoint (bosons). We treat (18) as an approximate result which is useful since it gives a single simple distribution formula for all exclusons, $\kappa \in [0, 1]$.

A connection between the two approaches requires a deeper study which can be made elsewhere.

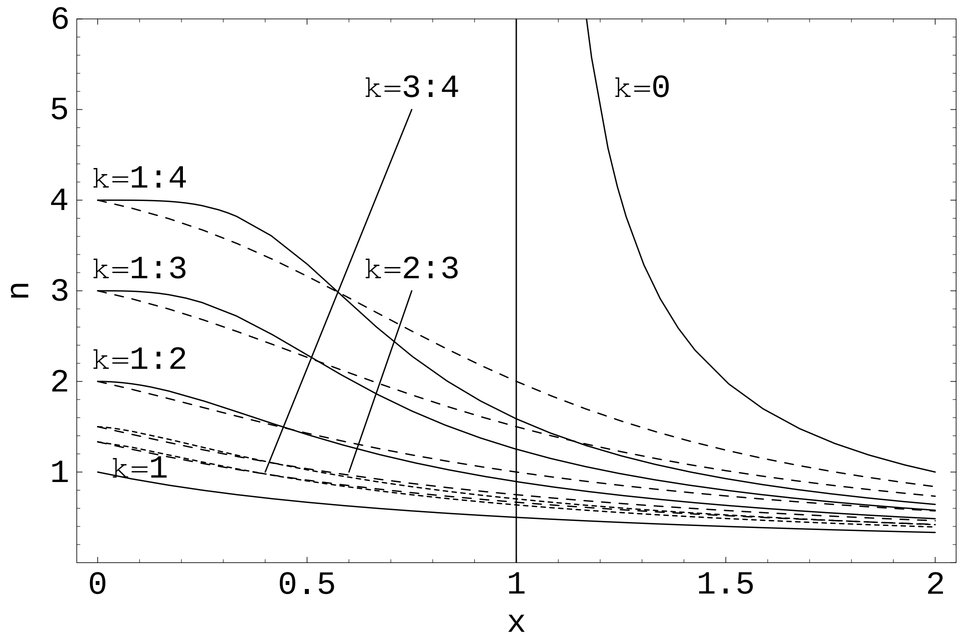

Figure 2: Statistical distribution n_i as a function of $x = \exp[(\varepsilon_i - \mu)/kT]$, for various κ; dashed lines represent the approximation (18) to exact solutions (solid lines) shown in Fig. 1.

4 Conclusions

(i) The derived statistical weight (3) and Haldane-Wu statistical weight (1) lead to the same distribution equation (7);

(ii) Haldane-Wu parameter λ acquires a physical meaning of an inverse of the maximal allowed occupation number in state, $\lambda = 1/q$, similar to the inverse of the statistical factor as shown by Wu [2];

(iii) Within fractional exclusion statistics, the generalized Pauli exclusion principle reads that a maximal allowed occupation number of identical particles in state is an integer, $q = 1, 2, 3, \ldots$, i.e. $n_i/z_i \leq 1/\lambda$ as formulated by Wu [2]. We stress that in our approach we use this principle as a basis to calculate statistical weight (3) rather than derive it *a posteriori* from the analysis of a statistical weight or distribution function;

(iv) While Haldane-Wu parameter λ is assumed to vary continuously, the statistical parameter $\kappa = 1/q$ runs over *discrete* set of values, $\kappa = 1, 1/2, 1/3, \ldots$ This may be an important difference since physically acceptable solutions of Eq. (7) may not exist for all values of $\kappa \in]0, 1[$, while $\kappa = 1, 1/2, 1/3, \ldots$ guarantees a polynomial structure of Eq. (7), with physically acceptable solutions;

(v) The equation (7), which defines statistical distribution of exclusons, $\kappa \in]0, 1[$, has a remarkable symmetry (10) which allows to interconnect solutions n_i for κ and $1 - \kappa$ due to Eq. (11).

In summary, we have shown that Haldane-Wu fractional exclusion statistics finds a natural combinatorial and physical interpretation in accord to Eq. (3), and presented exact solutions of Eq. (7).

References

[1] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).

[2] Y. S. Wu, Phys. Rev. Lett. **73**, 922 (1994).

[3] D. Bernard and Y. S. Wu, in Proc. 6th Nankai Workshop, eds. M. L. Ge and Y. S. Wu, World Scientific (1995); C. Nayak and F. Wilczek, Phys. Rev. Lett. **73**, 2740 (1994); Z. N. C. Ha, Nucl. Phys. **B 435**, 604 (1995); Y. Hatsugai, M. Kohmoto, T. Koma and Y. S. Wu, Phys. Rev. **B 54**, 5358 (1996); Y. Yu , H. X. Yang and Y. S. Wu, cond-mat/9911141; Y. S. Wu, Y. Yu , and H. X. Yang, Nucl. Phys. **B 604**, 551 (2001), cond-mat/0002063.

[4] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).